

Partial-wave Coulomb transition matrices for attractive interaction by Fock's method

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Leaning upon the Fock method of the stereographic projection of the three-dimensional momentum space onto the four-dimensional unit sphere the possibility of the analytical solving of the Lippmann-Schwinger integral equation for the partial wave two-body Coulomb transition matrix at the ground bound state energy has been studied. In this case new expressions for the partial p -, d - and f -wave two-body Coulomb transition matrices have been obtained in the simple analytical form. The developed approach can also be extended to determine analytically the partial wave Coulomb transition matrices at the energies of excited bound states.

Keywords: Partial wave Coulomb transition matrix; Lippmann-Schwinger equation; Fock method; Analytical solution

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1. Introduction

It is well known that the transition matrix of two interacting particles (with its elements both on and off the energy shell) encloses all the information on the system. In view of this, the transition matrix is an involved function of the variables of the initial and final momenta and the energy. In particular, the possible bound states of the system are responsible for formation in the off-shell t -matrix of the corresponding poles in the energy with the residues related with the wave functions of the system in these states. Similarly, providing the property of the unitarity of the scattering matrix, the states of the continuum spectrum at positive energies manifest themselves in existence in the transition matrix of a well-known singularity with a branch point and a cut along the positive energy axis.

A knowledge of the two-body off-shell Coulomb transition matrix is of great interest being necessary for description of the few-body atomic and nuclear systems. In particular, the Coulomb transition matrices are contained in the kernels of the Faddeev integral equations [1,2] for three particles, among which two or all particles are charged.

For charged particles, several representations of the two-body Coulomb transition matrix $t^C(E)$ (or the related Coulomb Green function $g^C(E)$) are known [3-12]. A considerable body of the earlier information on the Coulomb t -matrix is contained in the overview [13].

Of particular importance are the studies of the Coulomb transition matrix with the use of the symmetries of the Coulomb system both in the usual three-dimensional and Fock's four-dimensional [14] Euclidean spaces (Refs. [5] and [8]). In this way the expressions for the Coulomb transition matrix with the explicit removal of the singularities in the transfer-momentum variable $|\mathbf{k} - \mathbf{k}'|$ and the energy E have been

derived in the papers [10] (for the negative energy $E < 0$) and [11] (for the zeroth and positive energies $E \geq 0$).

This paper is devoted to the derivation of the partial wave two-body Coulomb transition matrices in the special case that the energy E is equal to the ground bound state energy of the complex. In Section 2 we start from the expression for three-dimensional Coulomb transition matrix at the negative energy obtained by us (in Ref.[10]). In Section 3 we formulate the method for solving the one-dimensional integral Lippmann-Schwinger equation for the partial wave Coulomb t -matrix at the negative energy $E < 0$. In Section 4 the general formula for the partial wave Coulomb t -matrix at the energy of the ground bound state $E = -b_1$ is derived. The analytical expressions for the p -, d - and f -wave components of the Coulomb t -matrix are given in Subsections 4.1 – 4.3. Section 5 is devoted to the concluding remarks.

2. Three-dimensional Coulomb transitions matrix at negative energy

The two-particle Coulomb Green operator $g^C(E) \equiv (E - h_0 - v^C)^{-1}$ and the transition operator $t^C(E)$, which describe the system of two charged particles 1 and 2, are related among themselves by the relation

$$g^C(E) = g_0(E) + g_0(E)t^C(E)g_0(E). \quad (1)$$

Here h_0 denotes the operator of the kinetic energy of the relative motion of the particles, $g_0(E) \equiv (E - h_0)^{-1}$ is the free Green operator, $v^C(r) = q_1q_2/r$ is the potential of the Coulomb interaction between the particles (q_i denotes the charge of the particle i , r is the distance between the particles). The quantity E denotes the total energy of the relative motion between the particles 1 and 2.

In the momentum representation the formula (1) has the form

$$\langle \mathbf{k} | g^C(E) | \mathbf{k}' \rangle = \langle \mathbf{k} | g_0(E) | \mathbf{k}' \rangle + \frac{1}{E - \frac{k^2}{2\mu}} \langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle \frac{1}{E - \frac{k'^2}{2\mu}}, \quad (2)$$

where \mathbf{k} and \mathbf{k}' are the relative momentum variables that correspond to the relative radius vectors \mathbf{r} and \mathbf{r}' in the coordinate space, $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass of the particles 1 and 2 ($\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, \mathbf{r}_i and m_i are the radius vector and the mass of the particle i),

$$\langle \mathbf{k} | g_0(E) | \mathbf{k}' \rangle = \frac{(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')}{E - \frac{k^2}{2\mu}}.$$

Three-dimensional Coulomb transition matrix $\langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle$ satisfies to the Lippmann-Schwinger integral equation

$$\langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle = \langle \mathbf{k} | v^C | \mathbf{k}' \rangle + \int \frac{d\mathbf{k}''}{(2\pi)^3} \langle \mathbf{k} | v^C | \mathbf{k}'' \rangle \frac{1}{E - \frac{k''^2}{2\mu}} \langle \mathbf{k}'' | t^C(E) | \mathbf{k}' \rangle, \quad (3)$$

where the matrix of the potential of the Coulomb interaction between the particles 1 and 2 is equal to

$$\langle \mathbf{k} | v^C | \mathbf{k}' \rangle = \frac{4\pi q_1 q_2}{|\mathbf{k} - \mathbf{k}'|^2}. \quad (4)$$

At the negative energy

$$E = -\frac{\hbar^2 \kappa^2}{2\mu} < 0 \quad (5)$$

the solution of the equation (3) has the form [10]

$$\begin{aligned} \langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle = & \frac{4\pi q_1 q_2 \kappa^2}{(k^2 + \kappa^2)(k'^2 + \kappa^2)} \left[\frac{1}{\sin^2 \frac{\omega}{2}} - 2\pi\gamma \frac{\cos \gamma \omega}{\sin \omega} - 2\gamma \frac{\sin 2\gamma \omega}{\sin \omega} \ln(\sin \frac{\omega}{2}) \right. \\ & + 4\pi\gamma c(\gamma) \cot \gamma\pi \frac{\sin \gamma \omega}{\sin \omega} + 2\gamma \frac{\cos \gamma \omega}{\sin \omega} \int_0^\omega d\varphi \sin \gamma \varphi \cot \frac{\varphi}{2} \\ & \left. + 4\gamma^2 \frac{\sin \gamma \omega}{\sin \omega} \int_\omega^\pi d\varphi \sin \gamma \varphi \ln(\sin \frac{\varphi}{2}) \right], \end{aligned} \quad (6)$$

where γ is the dimensionless quantity that denotes the Coulomb parameter

$$\gamma = \frac{\mu q_1 q_2}{\hbar^2 \kappa}, \quad (7)$$

\hbar is the reduced Planck's constant. The variable quantity ω in Eq.(6) refers to the angle between two 4-vectors $e \equiv (\mathbf{e}, e_0)$ and $e' \equiv (\mathbf{e}', e'_0)$ in a four-dimensional Euclidean space introduced by Fock [14],

$$\mathbf{e} = \frac{2\kappa \mathbf{k}}{\kappa^2 + k^2}, \quad e_0 = \frac{\kappa^2 - k^2}{\kappa^2 + k^2}; \quad \mathbf{e}' = \frac{2\kappa \mathbf{k}'}{\kappa^2 + k'^2}, \quad e'_0 = \frac{\kappa^2 - k'^2}{\kappa^2 + k'^2}; \quad (8)$$

$$\cos \omega = e \cdot e' = \mathbf{e} \cdot \mathbf{e}' + e_0 \cdot e'_0. \quad (9)$$

In this case, the vectors \mathbf{k} and \mathbf{k}' lie in the hyperplane, which is the stereographic projection of the unit sphere in the four-dimensional space, and the variable ω is defined by the expression

$$\sin^2 \frac{\omega}{2} = \frac{\kappa^2 |\mathbf{k} - \mathbf{k}'|^2}{(k^2 + \kappa^2)(k'^2 + \kappa^2)}, \quad 0 \leq \omega \leq \pi. \quad (10)$$

The function $c(\gamma)$ in Eq.(6) has the form

$$c(\gamma) = \frac{1}{2} \left(1 - \frac{1}{\pi} \int_0^\pi d\varphi \sin \gamma \varphi \cot \frac{\varphi}{2} \right) \quad (11)$$

or using the digamma functions

$$c(\gamma) = \theta(-\gamma) + \frac{\sin \gamma \pi}{2\pi} \left[\psi \left(\frac{|\gamma| + 1}{2} \right) - \psi \left(\frac{|\gamma|}{2} \right) - \frac{1}{|\gamma|} \right], \quad (12)$$

where $\psi(x) \equiv d/dx \ln \Gamma(x)$ and $\Gamma(x)$ are the digamma and gamma functions [15], and $\theta(x)$ is the usual step function

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases} \quad (13)$$

The first three terms of the expression in brackets (6) contain singularities in the transfer momentum:

$$|\mathbf{k} - \mathbf{k}'|^{-2}, \quad |\mathbf{k} - \mathbf{k}'|^{-1} \quad \text{and} \quad \ln \left\{ \kappa |\mathbf{k} - \mathbf{k}'| / (k^2 + \kappa^2)^{1/2} (k'^2 + \kappa^2)^{1/2} \right\}$$

respectively in the first, second and third terms. The other three terms in (6) are smooth functions of $|\mathbf{k} - \mathbf{k}'|$.

The singularities in the energy are contained in the fourth term of the expression (6). They arise only in the case of the attractive Coulomb interaction potential (with unlike electric charges, $\gamma < 0$) when the Coulomb parameter γ takes on the negative integer values ($\gamma = -n$, $n = 1, 2, 3, \dots$) and correspond to the bound states. At this points $\cot \gamma \pi$ has the pole singularities and the function $c(\gamma)$ is distinct from zero: $c(-n) = 1$. In the case of the repulsive Coulomb potential ($\gamma > 0$) for the positive integer value of γ the function vanishes, $c(n) = 0$, and the fourth term in (6) is finite:

$$\lim_{\gamma \rightarrow n} \pi c(\gamma) \cot \gamma \pi = \frac{(-1)^n}{2n} - \sum_{m=1}^n \frac{(-1)^m}{m} - \ln 2. \quad (14)$$

The fifth and sixth terms in the expression (6) are the smooth functions of the energy (or of the Coulomb parameter γ).

3. Partial wave decomposition of the Coulomb transitions matrix

For the Coulomb potential (4), which is local and spherically symmetric, the expansions of the matrix elements $\langle \mathbf{k} | v^C | \mathbf{k}' \rangle$ and $\langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle$ in partial waves are

$$\begin{aligned} \langle \mathbf{k} | v^C | \mathbf{k}' \rangle &= \sum_{l=0}^{\infty} (2l+1) v_l^C(k, k') P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \\ \langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle &= \sum_{l=0}^{\infty} (2l+1) t_l^C(k, k'; E) P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'). \end{aligned} \quad (15)$$

Here $P_l(x)$ is the Legendre polinomial, the quantity with the hat $\hat{\mathbf{k}}$ denotes the unit vector along \mathbf{k} and depends on the polar angles of \mathbf{k} , $\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' = \cos \theta$.

The elements of the partial Coulomb potential and transition matrices are

$$\begin{aligned} v_l^C(k, k') &= \frac{1}{2} \int_0^\pi d\theta \sin \theta P_l(\cos \theta) \langle \mathbf{k} | v^C | \mathbf{k}' \rangle, \\ t_l^C(k, k'; E) &= \frac{1}{2} \int_0^\pi d\theta \sin \theta P_l(\cos \theta) \langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle, \end{aligned} \quad (16)$$

The partial component of the Coulomb potential (4) is

$$v_l^C(k, k') = \frac{2\pi q_1 q_2}{kk'} Q_l \left(\frac{k^2 + k'^2}{2kk'} \right), \quad (17)$$

where the function $Q_l(x)$ is the Legendre function of the second kind [15]

$$Q_l(x) = \frac{1}{2} P_l(x) \ln \left(\frac{x+1}{x-1} \right) - W_{l-1}(x), \quad (18)$$

$$W_{-1}(x) = 0, \quad W_{l-1}(x) = \sum_{k=1}^l \frac{1}{k} P_{l-k}(x) P_{k-1}(x).$$

The partial wave component of the Coulomb transition matrix $t_l^C(k, k'; E)$ satisfies the one-dimensional integral equation that follows from (3)

$$t_l^C(k, k'; E) = v_l^C(k, k') + \int_0^\infty \frac{dk'' k''^2}{2\pi^2} v_l^C(k, k'') \frac{1}{E - \frac{k''^2}{2\mu}} t_l^C(k'', k'; E) , \quad (19)$$

We find the solution of the equation (19) in the case of the negative energy $E < 0$ by substituting the expression for the three-dimensional Coulomb transition matrix (6) into the formula for the partial transition matrix (16). Going from the integration variable θ (the angle between the three-dimensional vectors \mathbf{k} and \mathbf{k}') in (16) to the variable ω using the formulas following from the expression (10),

$$\cos \theta = \frac{\xi}{\eta} - \frac{1}{\eta} \sin^2 \frac{\omega}{2} = \frac{2\xi - 1 + \cos \omega}{2\eta} \quad \sin \theta \, d\theta = \frac{1}{2\eta} \sin \omega \, d\omega , \quad (20)$$

where

$$\xi = \frac{\kappa^2(k^2 + k'^2)}{(k^2 + \kappa^2)(k'^2 + \kappa^2)} , \quad \eta = \frac{2\kappa^2 k k'}{(k^2 + \kappa^2)(k'^2 + \kappa^2)} , \quad (21)$$

we obtain

$$t_l^C(k, k'; E) = \frac{1}{4\eta} \int_{\omega_0}^{\omega_\pi} d\omega \sin \omega \, P_l \left(\frac{2\xi - 1 + \cos \omega}{2\eta} \right) \langle \mathbf{k} | t^C(E) | \mathbf{k}' \rangle . \quad (22)$$

The integration limits in (22) are determined by the expressions

$$\omega_0 = 2 \arcsin \sqrt{\xi - \eta} , \quad \omega_\pi = 2 \arcsin \sqrt{\xi + \eta} , \quad (23)$$

in this case

$$\begin{aligned} \cos \omega_0 &= 1 - 2\xi + 2\eta , & \cos \omega_\pi &= 1 - 2\xi - 2\eta , \\ \sin \omega_0 &= 2\sqrt{\xi - \eta} \sqrt{1 - \xi + \eta} , & \sin \omega_\pi &= 2\sqrt{\xi + \eta} \sqrt{1 - \xi - \eta} . \end{aligned} \quad (24)$$

Using the expression for the three-dimensional transition matrix (6) we write the formula (22) for the partial wave Coulomb transition matrix $t_l^C(k, k'; E)$ at $E < 0$ in the form

$$\begin{aligned} t_l^C(k, k'; E) &= \frac{\pi q_1 q_2}{k k'} \int_{\omega_0}^{\omega_\pi} d\omega \, P_l \left(\frac{2\xi - 1 + \cos \omega}{2\eta} \right) \left\{ \cot \frac{\omega}{2} \right. \\ &\quad - \pi \gamma \cos \gamma \omega - \gamma \sin 2\gamma \omega \ln \left(\sin \frac{\omega}{2} \right) + 2\pi \gamma c(\gamma) \cot \gamma \pi \sin \gamma \omega \\ &\quad \left. + \gamma \cos \gamma \omega \int_0^\omega d\varphi \sin \gamma \varphi \cot \frac{\varphi}{2} + 2\gamma^2 \sin \gamma \omega \int_\omega^\pi d\varphi \sin \gamma \varphi \ln \left(\sin \frac{\varphi}{2} \right) \right\} . \end{aligned} \quad (25)$$

Notice that in the expression (25) the quantity γ by the definition (7) depends on κ or, in accordance with (5), on the energy E . The quantities ξ and η , like the limits ω_0 and ω_π , depend on all the variables k , k' and E of the partial transition matrix $t_l^C(k, k'; E)$.

4. Partial-wave Coulomb transition matrix at the ground bound state energy

The general expression for partial wave Coulomb transitions matrix $t_l^C(k, k'; E)$ (25) is simplified in special cases if the energy E takes the value of the bound (ground or excited) state of the two-particle system, $E = E_n$, where n is the principal quantum number (for each value n the angular momentum of the bound state l_0 can take all integer values from 0 to $n - 1$),

$$E_n = -\frac{\mu(q_1 q_2)^2}{2\hbar^2 n^2}, \quad n = 1, 2, \dots, \quad (26)$$

herewith according to (5) and (7),

$$\kappa = \kappa_n = \frac{\sqrt{-2\mu E_n}}{\hbar} = \frac{\mu |q_1 q_2|}{\hbar^2 n}, \quad \gamma = \gamma_n = -n. \quad (27)$$

In this work we restrict the study of the Coulomb transition matrix (25) at the energy of the ground bound state of the two-particle complex of the two-particle ($n = 1$):

$$E = E_1 = -b_1, \quad b_1 = \frac{\hbar^2 \kappa_1^2}{2\mu}, \quad \kappa = \kappa_1 \equiv \frac{\mu |q_1 q_2|}{\hbar^2}, \quad \gamma = \gamma_1 = -1. \quad (28)$$

According to (25) and the relations of orthogonality and normalization for the Legendre polynomials, we find

$$\int_{\omega_0}^{\omega_\pi} d\omega \sin \omega P_l \left(\frac{2\xi - 1 + \cos \omega}{2\eta} \right) = 2\eta \int_0^\pi d\theta \sin \theta P_l(\cos \theta) = 4\eta \delta_{l0}, \quad (29)$$

and hence only s-wave partial component $t_0^C(k, k'; E)$ has the pole singularity in energy at the point $E = E_1 = -b_1$ (that locates in the fourth term) of the expression in braces (25). All higher partial components $t_l^C(k, k'; E)$ with $l > 0$ are non-singular at this point.

The expression for partial Coulomb transition matrices with $l > 0$ at the energy of the ground bound state of the two-particle complex (28) we obtain using the expression (25) with $\gamma \rightarrow -1$. Herewith, we evaluate the indeterminate form of the type $\frac{0}{0}$, which comes into being after the integration with respect to ω with the fourth term of the expression in braces (25) at $\gamma \rightarrow -1$, according to the l'Hospital rule:

$$\begin{aligned} & \lim_{\gamma \rightarrow -1} \frac{2\pi\gamma c(\gamma) \int_{\omega_0}^{\omega_\pi} d\omega \sin \gamma\omega P_l \left(\frac{2\xi - 1 + \cos \omega}{2\eta} \right)}{\tan \gamma\pi} \\ & \rightarrow -2 \int_{\omega_0}^{\omega_\pi} d\omega \omega \cos \omega P_l \left(\frac{2\xi - 1 + \cos \omega}{2\eta} \right). \end{aligned} \quad (30)$$

At the energy of the ground bound state ($\gamma = -1$) the integrals over φ in the fifth and sixth terms of the expression (25) are equal to

$$\int_0^\omega d\varphi \sin \varphi \cot \frac{\varphi}{2} = \omega + \sin \omega,$$

$$\int_{\omega}^{\pi} d\varphi \sin \varphi \ln \left(\sin \frac{\varphi}{2} \right) = -\cos^2 \frac{\omega}{2} - 2 \sin^2 \frac{\omega}{2} \ln \left(\sin \frac{\omega}{2} \right) . \quad (31)$$

In this case the expression in braces of the general expression for the partial Coulomb transition matrix (25) acquires the form

$$\cot \frac{\omega}{2} + \pi \cos \omega - \omega \cos \omega - \sin \omega - 2 \sin \omega \ln \left(\sin \frac{\omega}{2} \right) . \quad (32)$$

It should be noted that the fourth term in the expression (32) does not contribute to all partial components $t_l^C(k, k'; -b_1)$ with $l > 0$ in view of the orthogonality relation (29).

As a result the expression (25) for the partial wave Coulomb transition matrix (with $l = 1, 2, 3, \dots$) at the ground bound state energy ($\gamma = -1$) is simplified to the form

$$t_l^C(k, k'; -b_1) = \frac{\pi q_1 q_2}{k k'} \int_{\omega_{01}}^{\omega_{\pi 1}} d\omega P_l \left(\frac{2\xi_1 - 1 + \cos \omega}{2\eta_1} \right) \cdot \left\{ \cot \frac{\omega}{2} + \pi \cos \omega - \omega \cos \omega - 2 \sin \omega \ln \left(\sin \frac{\omega}{2} \right) \right\} , \quad (33)$$

where the quantities $\xi_1, \eta_1, \omega_{01}$ and $\omega_{\pi 1}$ are determined by the expressions for ξ, η, ω_0 and ω_{π} according to the definitions (21) and (23) taken at the value $\kappa = \kappa_1$ that corresponds to the energy of the ground bound state (28),

$$\xi_1 = \frac{\kappa_1^2(k^2 + k'^2)}{(k^2 + \kappa_1^2)(k'^2 + \kappa_1^2)} , \quad \eta_1 = \frac{2\kappa_1^2 k k'}{(k^2 + \kappa_1^2)(k'^2 + \kappa_1^2)} ,$$

$$\omega_{01} = 2 \arcsin \sqrt{\xi_1 - \eta_1} , \quad \omega_{\pi 1} = 2 \arcsin \sqrt{\xi_1 + \eta_1} , \quad (34)$$

Notice that taking into account only the first term in the braces of the expression (33) corresponds to the Born approximation for the partial Coulomb transition matrix

$$t_l^{C,B}(k, k'; -b_1) = v_l^C(k, k') \quad (35)$$

that is determined by the formulas (17) and (18).

Hereafter, using the spectroscopic notation, we designate the low partial waves corresponding to $l = 0, 1, 2, 3, \dots$ as s, p, d, f, \dots waves.

4.1. Partial p -wave Coulomb transition matrix at the energy $E = -b_1$

The formula for the p -wave Coulomb t -matrix at the ground bound state energy follows immediately after integration in the expression (33) with $l = 1$. The separate four terms in the braces of the expression (33) make the following contributions to the p -wave t -matrix:

$$t_1^C(k, k'; -b_1) = \frac{\pi q_1 q_2}{k k'} \{ \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 \} , \quad (36)$$

where

$$\mathcal{P}_1 = \frac{\xi_1}{\eta_1} \ln \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} - 2 ,$$

$$\mathcal{P}_2 = \frac{\pi}{4\eta_1} [\omega_{\pi 1} - \omega_{01} - \sin(\omega_{\pi 1} - \omega_{01})] ,$$

$$\begin{aligned}\mathcal{P}_3 &= 2\xi_1 - 1 + \frac{1}{\eta_1} \left[-\frac{1}{8} (\omega_{\pi 1}^2 - \omega_{01}^2) \right. \\ &\quad \left. + \frac{1}{4} (\omega_{\pi 1} \sin \omega_{\pi 1} \cos \omega_{01} - \omega_{01} \cos \omega_{\pi 1} \sin \omega_{01}) \right] , \\ \mathcal{P}_4 &= 2\xi_1 - \frac{1}{\eta_1} (\xi_1^2 - \eta_1^2) \ln \frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} .\end{aligned}\tag{37}$$

The first term in (36) is the Born approximation for the p -wave Coulomb t -matrix

$$t_1^{C,B}(k, k'; -b_1) = v_1^C(k, k') = \frac{\pi q_1 q_2}{k k'} \mathcal{P}_1 .\tag{38}$$

The first and fourth addends in (36) contain the logarithmic functions.

Adding up in (36) all four terms (37) we obtain the following expression for the p -wave Coulomb transition matrix at $E = -b_1$:

$$\begin{aligned}t_1^C(k, k'; -b_1) &= \frac{\pi q_1 q_2}{k k'} \left\{ 4\xi_1 - 3 + \frac{1}{\eta_1} \left[(\xi_1 - \xi_1^2 + \eta_1^2) \ln \left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{8} (\omega_{\pi 1} - \omega_{01}) (2\pi - \omega_{\pi 1} - \omega_{01}) - \frac{1}{4} (\pi - \omega_{\pi 1}) \sin \omega_{\pi 1} \cos \omega_{01} \right. \right. \\ &\quad \left. \left. + \frac{1}{4} (\pi - \omega_{01}) \cos \omega_{\pi 1} \sin \omega_{01} \right] \right\} .\end{aligned}\tag{39}$$

4.2. Partial d -wave Coulomb transition matrix at the energy $E = -b_1$

In a similar way, performing integration in the expression (33) with $l = 2$, we obtain the formula for the d -wave Coulomb t -matrix at the ground bound state energy:

$$t_1^C(k, k'; -b_1) = \frac{\pi q_1 q_2}{k k'} \{ \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 \} .\tag{40}$$

The separate terms in (40) correspond to the addends in the braces of the expression (33) for the d -wave component of the t -matrix,

$$\begin{aligned}\mathcal{D}_1 &= \left[\frac{3}{2} \left(\frac{\xi_1}{\eta_1} \right)^2 - \frac{1}{2} \right] \ln \left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) - 3 \frac{\xi_1}{\eta_1} , \\ \mathcal{D}_2 &= \frac{1}{\eta_1^2} \left\{ \frac{3\pi}{8} (2\xi_1 - 1) (\omega_{\pi 1} - \omega_{01}) + \frac{\pi}{8} [X_2(\xi_1, \eta_1) \sin \omega_{\pi 1} - X_2(\xi_1, -\eta_1) \sin \omega_{01}] \right\} , \\ \mathcal{D}_3 &= \frac{1}{\eta_1} \left(2\xi_1^2 - 2\xi_1 - \frac{4}{3}\eta_1^2 + \frac{3}{2} \right) - \frac{1}{\eta_1^2} \left[\frac{3}{16} (2\xi_1 - 1) (\omega_{\pi 1}^2 - \omega_{01}^2) \right. \\ &\quad \left. + \frac{1}{8} X_2(\xi_1, \eta_1) \omega_{\pi 1} \sin \omega_{\pi 1} - \frac{1}{8} X_2(\xi_1, -\eta_1) \omega_{01} \sin \omega_{01} \right] , \\ \mathcal{D}_4 &= \frac{1}{\eta_1} \left(2\xi_1^3 - \frac{4}{3}\eta_1^3 \right) - \frac{1}{\eta_1^2} (\xi_1^3 - \xi_1 \eta_1^2) \ln \left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) ,\end{aligned}\tag{41}$$

Herewith the notation $X_2(\xi_1, \eta_1)$ in the expressions for \mathcal{D}_2 and \mathcal{D}_3 has the form

$$X_2(\xi_1, \eta_1) = 4\xi_1^2 - 4\xi_1 - 4\xi_1\eta_1 + 2\eta_1 + 3 .\tag{42}$$

Using the formulas (24), the expressions for $X_2(\xi_1, \eta_1)$ and $X_2(\xi_1, -\eta_1)$ can be written in the form

$$X_2(\xi_1, \eta_1) = -(2\xi_1 - 1) \cos \omega_{01} + 2, \quad X_2(\xi_1, -\eta_1) = -(2\xi_1 - 1) \cos \omega_{\pi 1} + 2. \quad (43)$$

As in the preceding case of the p -wave component of the t -matrix, the first term in (40) is the Born approximation for the d -wave component,

$$t_2^{C,B}(k, k'; -b_1) = v_2^C(k, k') = \frac{\pi q_1 q_2}{kk'} \mathcal{D}_1. \quad (44)$$

Adding up in (40) all four terms (41) and taking into consideration the relations (43), we obtain the formula for the d -wave Coulomb transition matrix at $E = -b_1$:

$$\begin{aligned} t_2^C(k, k'; -b_1) = & \frac{\pi q_1 q_2}{kk'} \left\{ \frac{1}{\eta_1} \left(4\xi_1^2 - 5\xi_1 - \frac{8}{3}\eta_1^2 + \frac{3}{2} \right) \right. \\ & + \frac{1}{\eta_1^2} \left[\left(-\xi_1^3 + \frac{3}{2}\xi_1^2 + \xi_1\eta_1^2 - \frac{1}{2}\eta_1^2 \right) \ln \left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) + \frac{3}{16} (2\xi_1 - 1) (\omega_{\pi 1} - \omega_{01}) (2\pi - \omega_{\pi 1} - \omega_{01}) \right. \\ & \left. \left. + \frac{1}{8} (\pi - \omega_{\pi 1}) X_2(\xi_1, \eta_1) \sin \omega_{\pi 1} - \frac{1}{8} (\pi - \omega_{01}) X_2(\xi_1, -\eta_1) \sin \omega_{01} \right] \right\}, \end{aligned} \quad (45)$$

4.3. Partial f -wave Coulomb transition matrix at the energy $E = -b_1$

Likewise, integrating in the formula (33) with $l = 3$, we determine the contributions from the individual terms in the braces to the f -wave Coulomb transition matrix:

$$t_3^C(k, k'; -b_1) = \frac{\pi q_1 q_2}{kk'} \{ \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 \}, \quad (46)$$

where

$$\begin{aligned} \mathcal{F}_1 = & \frac{1}{\eta_1^2} \left(-5\xi_1^2 + \frac{4}{3}\eta_1^2 \right) + \frac{1}{2\eta_1^3} \left(5\xi_1^3 - 3\xi_1\eta_1^2 \right) \ln \left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right), \\ \mathcal{F}_2 = & \frac{\pi}{16\eta_1^3} \left[6 \left(5\xi_1^2 - 5\xi_1 - \eta_1^2 + \frac{25}{16} \right) (\omega_{\pi 1} - \omega_{01}) \right. \\ & \left. + X_3(\xi_1, \eta_1) \sin \omega_{\pi 1} - X_3(\xi_1, -\eta_1) \sin \omega_{01} \right], \\ \mathcal{F}_3 = & \frac{1}{\eta_1^2} \left(\frac{5}{2}\xi_1^3 - \frac{15}{4}\xi_1^2 + \frac{95}{16}\xi_1 - \frac{13}{6}\xi_1\eta_1^2 + \frac{13}{12}\eta_1^2 - \frac{75}{32} \right) \\ & + \frac{1}{16\eta_1^3} \left[-3 \left(5\xi_1^2 - 5\xi_1 - \eta_1^2 + \frac{25}{16} \right) (\omega_{\pi 1}^2 - \omega_{01}^2) \right. \\ & \left. - \omega_{\pi 1} X_3(\xi_1, \eta_1) \sin \omega_{\pi 1} + \omega_{01} X_3(\xi_1, -\eta_1) \sin \omega_{01} \right], \\ \mathcal{F}_4 = & \frac{1}{\eta_1^2} \left(\frac{5}{2}\xi_1^3 - \frac{13}{6}\xi_1\eta_1^2 \right) + \frac{1}{\eta_1^3} \left(-\frac{5}{4}\xi_1^4 + \frac{3}{2}\xi_1^2\eta_1^2 - \frac{1}{4}\eta_1^4 \right) \ln \left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right), \end{aligned} \quad (47)$$

herewith the notation $X_2(\xi_1, \eta_1)$ in the expressions for \mathcal{F}_2 and \mathcal{F}_3 has the form

$$X_3(\xi_1, \eta_1) = 10\xi_1^3 - 15\xi_1^2 + \frac{95}{4}\xi_1 - 10\xi_1^2\eta_1 + 10\xi_1\eta_1 - 2\xi_1\eta_1^2 - \frac{25}{4}\eta_1 + \eta_1^2 + 2\eta_1^3 - \frac{75}{8}. \quad (48)$$

Using the formulas (24), the expressions for $X_3(\xi_1, \eta_1)$ and $X_3(\xi_1, -\eta_1)$ can be written in the form

$$\begin{aligned} X_3(\xi_1, \eta_1) &= -\left(5\xi_1^2 - 5\xi_1 - \eta_1^2 + \frac{75}{8}\right) \cos \omega_{01} + \frac{25}{2}\eta_1, \\ X_3(\xi_1, -\eta_1) &= -\left(5\xi_1^2 - 5\xi_1 - \eta_1^2 + \frac{75}{8}\right) \cos \omega_{\pi 1} - \frac{25}{2}\eta_1. \end{aligned} \quad (49)$$

The first addend in (46) is the Born approximation for the f -wave Coulomb t -matrix

$$t_3^{C,B}(k, k'; -b_1) = v_3^C(k, k') = \frac{\pi q_1 q_2}{k k'} \mathcal{F}_1. \quad (50)$$

Adding up in (46) all four terms (47) and taking into consideration the relations (49), we derive the formula for the f -wave Coulomb transition matrix at the ground bound state energy $E = -b_1$:

$$\begin{aligned} t_3^C(k, k'; -b_1) &= \frac{\pi q_1 q_2}{k k'} \left\{ \frac{1}{\eta_1^2} \left(5\xi_1^3 - \frac{35}{4}\xi_1^2 + \frac{95}{16}\xi_1 - \frac{13}{3}\xi_1\eta_1^2 + \frac{29}{12}\eta_1^2 - \frac{75}{32} \right) \right. \\ &\quad + \frac{1}{\eta_1^3} \left[\left(-\frac{5}{4}\xi_1^4 + \frac{5}{2}\xi_1^3 + \frac{3}{2}\xi_1^2\eta_1^2 - \frac{3}{2}\xi_1\eta_1^2 - \frac{1}{4}\eta_1^4 \right) \ln \left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1} \right) \right. \\ &\quad \left. + \frac{3}{16} \left(5\xi_1^2 - 5\xi_1 - \eta_1^2 + \frac{25}{16} \right) (\omega_{\pi 1} - \omega_{01}) (2\pi - \omega_{\pi 1} - \omega_{01}) \right. \\ &\quad \left. \left. + \frac{1}{16} X_3(\xi_1, \eta_1) (\pi - \omega_{\pi 1}) \sin \omega_{\pi 1} - \frac{1}{16} X_3(\xi_1, -\eta_1) (\pi - \omega_{01}) \sin \omega_{01} \right] \right\}, \end{aligned} \quad (51)$$

5. Concluding remarks

In this work we have developed the method of the analytical solving the integral Lippmann-Schwinger equation for the partial wave off-shell two-body Coulomb transition matrix at the energy of the ground bound state, $t_l^C(k, k'; -b_1)$ ($l = 1, 2, 3, \dots$), using the Fock's dynamical symmetry of the system with the Coulomb interaction described by the four-dimensional rotation group $SO(4)$.

Specifically, a knowledge of the partial-wave Coulomb transition matrix at the bound state energy is necessary to determine the electric multipole polarizabilities of the hydrogen-like atoms.

Restricting our attention to the hydrogen-like atom with an infinite-mass nucleus we write the expression for the electric 2^λ -pole polarizabilities of the two-body bound complex $\alpha_{E\lambda}$ ($\lambda = 1, 2, 3, \dots$) in terms of the wave function, its corresponding derivatives and the partial wave transition matrix t_λ at the energy of the bound state $E = -b_1$ (Ref. 16):

$$\begin{aligned} \alpha_{E\lambda} &= \frac{2}{(2\lambda + 1)\pi^2} \frac{m_1 q_1^2}{\hbar^2} \left\{ \int_0^\infty dk k^2 \frac{|\varphi_\lambda(k)|^2}{k^2 + \kappa_1^2} \right. \\ &\quad \left. - \frac{m_1}{\pi^2 \hbar^2} \int_0^\infty dk k^2 \int_0^\infty dk' k'^2 \frac{\varphi_\lambda^*(k) t_\lambda(k, k'; -b_1) \varphi_\lambda(k')}{(k^2 + \kappa_1^2)(k'^2 + \kappa_1^2)} \right\}, \end{aligned} \quad (52)$$

where m_1 and q_1 are the mass and the charge of the electron,

$$\varphi_\lambda(k) = (-1)^\lambda k^\lambda \left[\left(\frac{1}{k} \frac{d}{dk} \right)^\lambda \psi(k) \right], \quad (53)$$

$\psi(k)$ is the wave function of the S -wave ground bound state of the atom in the momentum space.

Introducing into consideration a so-called *off-shell-energy scattering function*

$$\phi_\lambda(k) = -\frac{m_1}{\pi^2 \hbar^2} \int_0^\infty dk' k'^2 t_\lambda^C(k, k'; -b_1) \frac{1}{k'^2 + \kappa_1^2} \varphi_\lambda(k') \quad (54)$$

the integral equation for which follows from the Lippmann-Schwinger equation for the partial transition matrix at $E = -b_1$ (19),

$$\phi_\lambda(k) = f_\lambda(k) - \frac{m_1}{\pi^2 \hbar^2} \int_0^\infty dk' k'^2 v_\lambda^C(k, k') \frac{1}{k'^2 + \kappa_1^2} \phi_\lambda(k'), \quad (55)$$

where the free term is determined by the expression

$$f_\lambda(k) = -\frac{m_1}{\pi^2 \hbar^2} \int_0^\infty dk' k'^2 v_\lambda^C(k, k') \frac{1}{k'^2 + \kappa_1^2} \varphi_\lambda(k'), \quad (56)$$

we write the formula (52) as

$$\alpha_{E\lambda} = \frac{2}{(2\lambda + 1)\pi^2} \frac{m_1 q_1^2}{\hbar^2} \int_0^\infty dk k^2 \frac{\varphi_\lambda^*(k) [\varphi_\lambda(k) + \phi_\lambda(k)]}{k^2 + \kappa_1^2}. \quad (57)$$

In the preceding paper [17] the integral equation (55) for the hydrogen-like atom has been analytically solved. It has also been shown that the formula (57) reproduces the analytical result for the multipole electric polarizability derived by Dalgarno and Lewis in Ref. [18].

The method of the derivation of the analytical expression for the partial wave Coulomb transition matrices developed in this work can be generalized for determination of the transition matrix at the excited energies, $t_l^C(k, k'; -b_n)$ with $n > 1$, the knowledge of which, in particular, makes it possible to obtain information on the electric polarizabilities of the hydrogen-like atom in the excited states.

In nuclear physics, the direct approach using the off-shell transition matrix has been applied in [19] to calculate the electric multipole polarizabilities of the deuteron.

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